Gray Code

A Gray code is an encoding of numbers so that adjacent numbers have a single digit differing by 1. The term Gray code is often used to refer to a "reflected" code, or more specifically still, the binary reflected Gray code.

To convert a binary number $d_1 \, d_2 \cdots d_{n-1} \, d_n$ to its corresponding binary reflected Gray code, start at the right with the digit d_n (the *n*th, or last, digit). If the d_{n-1} is 1, replace d_n by $1 - d_n$; otherwise, leave it unchanged. Then proceed to d_{n-1}. Continue up to the first digit d_1, which is kept the same since d_0 is assumed to be a 0. The resulting number $g_1 \, g_2 \cdots g_{n-1} \, g_n$ is the reflected binary Gray code.

To convert a binary reflected Gray code $g_1 \, g_2 \cdots g_{n-1} \, g_n$ to a binary number, start again with the *n*th digit, and compute

$$\Sigma_n \equiv \sum_{i=1}^{n-1} g_i \pmod 2.$$

If Σ_n is 1, replace g_n by $1 - g_n$; otherwise, leave it the unchanged. Next compute

$$\Sigma_{n-1} \equiv \sum_{i=1}^{n-2} g_i \pmod 2,$$

and so on. The resulting number $d_1 \, d_2 \cdots d_{n-1} \, d_n$ is the binary number corresponding to the initial binary reflected Gray code.

The code is called reflected because it can be generated in the following manner. Take the Gray code 0, 1. Write it forwards, then backwards: 0, 1, 1, 0. Then prepend 0s to the first half and 1s to the second half: 00, 01, 11, 10. Continuing, write 00, 01, 11, 10, 10, 11, 01, 00 to obtain: 000, 001, 011, 010, 110, 111, 101, 100, ... (Sloane's A014550). Each iteration therefore doubles the number of codes.

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The plots above show the binary representation of the first 255 (top figure) and first 511 (bottom figure) Gray codes. The Gray codes corresponding to the first few nonnegative integers are given in the following table.

0	0	20	11110	40	111100
1	1	21	11111	41	111101
2	11	22	11101	42	111111
3	10	23	11100	43	111110
4	110	24	10100	44	111010
5	111	25	10101	45	111011
6	101	26	10111	46	111001
7	100	27	10110	47	111000
8	1100	28	10010	48	101000
9	1101	29	10011	49	101001
10	1111	30	10001	50	101011
11	1110	31	10000	51	101010
12	1010	32	110000	52	101110
13	1011	33	110001	53	101111
14	1001	34	110011	54	101101
15	1000	35	110010	55	101100
16	11000	36	110110	56	100100
17	11001	37	110111	57	100101
18	11011	38	110101	58	100111
19	11010	39	110100	59	100110

The binary reflected Gray code is closely related to the solutions of the towers of Hanoi and baguenaudier, as well as to Hamiltonian circuits of hypercube graphs (including direction reversals; Skiena 1990, p. 149).

SEE ALSO: Baguenaudier, Binary, Hilbert Curve, Ryser Formula, Thue-Morse Sequence, Towers of Hanoi

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Gray code

From Wikipedia, the free encyclopedia.
(Redirected from Gray coding)

A **Gray code** is a binary numeral system where two successive values differ in only one digit, originally designed to prevent spurious output from electromechanical switches.

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Motivation

Many devices indicate position by closing and opening switches. If that device uses natural binary codes, these two positions would be right next to each other:

```
...
011
100
...
```

The problem with natural binary codes is that, with real (mechanical) switches, it is very unlikely that switches will change states exactly in synchrony. In the transition between the two states shown above, all three switches change state. In the brief period while all are changing, the switches will read some spurious position. Even without keybounce, the transition might look like 011 -- 001 -- 101 -- 100. When the switches appear to be in position 001, the observer cannot tell if that is the "real" position 001, or a transitional state between two other positions.

A Gray code solves this problem by changing only one switch at a time, so there is never any ambiguity of position:

```
0 000
1 001
2 011
3 010
4 110
5 111
6 101
7 100
```

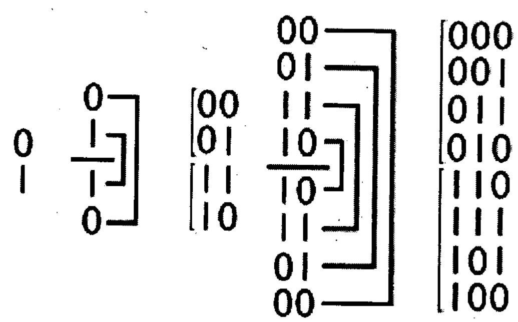
Notice that state 7 can roll over to state 0 with only one switch change. This is called the "cyclic" property of a Gray code.

More formally, a **Gray code** is a code assigning to each of a contiguous set of integers, or to each member of a circular list, a word of symbols such that each two adjacent code words differ by one symbol. These codes are also known as *single-distance codes*, reflecting the Hamming distance of 1 between adjacent codes. There can be more than one Gray code for a given word length, but the term was first applied to a particular binary code for the non-negative integers, the *binary-reflected Gray code*, or BRGC, the three-bit version of which is shown above.

Construction

The binary-reflected Gray code for n bits can be generated recursively by prefixing a binary 0 to the Gray code for $n-1$ bits, then prefixing a binary 1 to the reflected (i.e. listed in reverse order) Gray code for $n-1$ bits. The base case, for $n=1$ bit, is the most basic Gray code, $G = \{0, 1\}$. The BRGC may also be constructed iteratively.

Here are the first few steps of the above-mentioned reflect-and-prepend method:



Programming algorithms

Here is one algorithm in pseudocode to convert natural binary codes to Gray code (encode):

```
Let B[n:0] the array of bits in the usual binary representation
Let G[n:0] the array of bits in Gray code
  G[n]=B[n]
  for i=n-1 down to i=0 {
    G[i]=B[i+1] XOR B[i]
  }
```

or in C:

```
unsigned int grayencode(unsigned int g) {
  return(g^g>>1);
}
```

Here is one pseudocode algorithm to convert Gray code to natural binary codes (decode):

```
  B[n]=G[n]
  for i=n-1 down to i=0 {
    B[i]=B[i+1] XOR G[i]
  }
```

or in C:

```
unsigned int graydecode(unsigned int b) {
  b^=b>>1;
  b^=b>>2;
  b^=b>>4;
  b^=b>>8;
  return(b^(b>>16));
}
```

History and practical application

Gray codes (not so named) were applied to mathematical puzzles before they became known to engineers. The French engineer Émile Baudot used Gray codes in telegraphy in 1878. He received the French Legion of Honor medal for his work.

A vacuum tube using Gray encoding was patented (see below) in 1953 by Frank Gray, a researcher at Bell Labs, who gave his name to the codes. The use of his eponymous codes that Gray was most interested in was to minimize the effect of error in the transmission of digital signals; his codes are still used today for this purpose, and others.

Gray codes are used in angle-measuring devices in preference to straightforward binary encoding. This avoids the possibility that, when several bits change in the binary representation of an angle, a misread could result from some of the bits changing before others. This application benefits from the cyclic nature of Gray codes, because the first and last values of the sequence differ by only one bit.

The binary-reflected Gray code can also be used to serve as a solution guide for the Tower of Hanoi problem. A detailed method may be found here (*http://occawlonline.pearsoned.com/bookbind/pubbooks/miller2_awl/chapter4/essay1/deluxe-content.html#tower*).

Due to the Hamming distance properties of Gray codes, they are sometimes used in Genetic Algorithms.

Special types of Gray codes

n-ary Gray code

There are many specialized types of Gray codes other than the binary-reflected Gray code. One such type of Gray code is the **n-ary Gray code**, also known as a **non-Boolean Gray code**. As the name implies, this type of Gray code uses non-Boolean values in its encodings. For example, a 3-ary (ternary) Gray code would use the values {0, 1, 2}. The *(n,k)-Gray code* is the *n*-ary Gray code with *k* digits [5]. The sequence of elements in the *(3,2)-*Gray code is: *{00, 01, 02, 12, 11, 10, 20, 21, 22}*. It is important to note that an *(n,k)*-Gray code with odd *n* lacks the cyclic property of a binary Gray code; it can be observed that in going from the last element in the sequence, *22*, and wrapping around to the first element in the sequence, *00*, two digits change, unlike in a binary Gray code, in which only one digit would change. An *(n,k)*-Gray code with even *n*, however, retains the cyclic property of the binary Gray code. The *(n,k)*-Gray code may be constructed recursively, as the BRGC, or may be constructed iteratively. A pseudocode algorithm to iteratively generate the *(n,k)*-Gray code based off the work of Dah-Jyu Guan [5] is presented:

```
int n[k+1]; // stores the maximum for each digit
int g[k+1]; // stores the Gray code
int u[k+1]; // stores +1 or -1 for each element
int i, j;

// initialize values
for(i = 0; i <= k; i++) {
        g[i] = 0;
        u[i] = 1;
        n[i] = n;
}

// generate codes
while(g[k] == 0) {
```

```
        i = 0;
        j = g[0] + u[0];
        while((j >= n[i]) || (j < 0)) {
                u[i] = -u[i];
                i++;
                j = g[i] + u[i];
        }
        g[i] = j;
}
// g[i] now holds the (n,k)-Gray code
```

Beckett-Gray code

Another interesting type of Gray code is the **Beckett-Gray code**. The Beckett-Gray code is named after Samuel Beckett, a British playwright especially interested in symmetry. One of his plays, "Quad", was divided into sixteen time periods. At the end of each time period, Beckett wished to have one of the four actors either entering or exiting the stage; he wished the play to begin and end with an empty stage; and he wished each subset of actors to appear on stage exactly once [4]. Clearly, this meant the actors on stage could be represented by a 4-bit binary Gray code. Beckett placed an additional restriction on the scripting, however: he wished the actors to enter and exit such that the actor who had been on stage the longest would always be the one to exit. The actors could then be represented by a first in, first out queue data structure, so that the first actor to exit when a dequeue is called for is always the first actor which was enqueued into the structure [4]. Beckett was unable to find a Beckett-Gray code for his play, and indeed, an exhaustive listing of all possible sequences reveals that no such code exists for $n = 4$. Computer scientists interested in the mathematics behind Beckett-Gray codes have found these codes very difficult to work with. It is today known that codes exist for $n = \{2, 5, 6\}$ and they do not exist for $n = \{3, 4\}$. The search space for $n = 6$ is so large that it has not been exhaustively searched and several hundred thousand Beckett-Gray codes for $n = 6$ are known; the search space for $n = 7$ is so large that only a non-cyclic Beckett-Gray code (and therefore not technically of the kind originally proposed by Beckett) was found after several months of computing time [4].

Single-track Gray code

The **single-track Gray code** was originally defined by Hiltgen, Paterson and Brandestini. The STGC is a cyclical list of P unique binary encodings of length n such that two consecutive words differ in exactly one position, and when the list is examined as a $P \times n$ matrix, each column is a cyclic shift of the first column [3]. An $n = 5$ STGC is reproduced here:

```
10000   01000   00100   00010   00001
10100   01010   00101   10010   01001
11100   01110   00111   10011   11001
11110   01111   10111   11011   11101
11010   01101   10110   01011   10101
11000   01100   00110   00011   10001
```

Note that each column is a cyclic shift of the first column, and if each entry is read down each column and from the bottom entry of one column to the top

of the next, only one bit changes [7]. The STGC is useful to measure the absolute angular position of a rotating wheel by encoding (optically) the code words on *n* concentric tracks [3]. The single-track nature is useful in the fabrication of these wheels, as only one track design is needed, thus reducing their cost; and the Gray code nature is useful, as only one track will change at any one time, so the uncertainty during a transition between two discrete states will only be plus or minus one degree of angular measurement the device is capable of resolving [1].

See also

- Binary-coded decimal
- Linear feedback shift register

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External links

- NIST Dictionary of Algorithms and Data Structures: Gray code (*http://www.nist.gov/dads/HTML/graycode.html*)
- *Numerical Recipes in C*, section 20.2 (*http://lib-www.lanl.gov/numerical/bookcpdf/c20-2.pdf*) describing Gray codes in detail (ISBN 0521431085)
- Hitch Hiker's Guide to Evolutionary Computation, Q21: What are Gray codes, and why are they used? (*http://www.cs.bham.ac.uk/Mirrors/ftp.de.uu.net/EC/clife/www/Q21.htm*), including C code to convert between binary and BRGC

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